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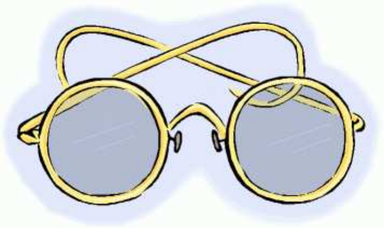
(Sections 1.1-1.2 and Section 5.1A)

Algebra vs. Geometry



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Pictures of Numbers: The Number Line

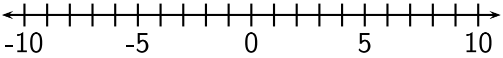
Every real number corresponds to a point on the number line. Every point on the number line corresponds to a real number.

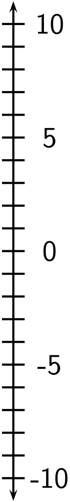
Traditionally, a smaller number appears to the ______________ of a larger number on a horizontal number line. Traditionally, a smaller number appears ______________ a larger number on a vertical number line.

Pictures of Numbers: The Number Line

Some things to know about the pictures you can draw with number lines:

1. Points that are shaded correspond to numbers that you want to include.
2. Points that are not shaded correspond to numbers that you do not want to include.
3. •, [, or] means that you include the number.
4. ○, (,) means that you do not include the number.





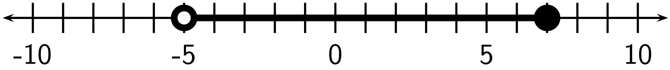
Pictures of Numbers: The Number Line

Your textbook allows you to use parentheses and brackets in the pictures that you draw on number lines. You are welcome to do this, but you also need to know the appropriate use for $\bullet$ and $\circ$ on the number line. These are more useful when we move into higher dimensions, and your instructor is likely to use them in class and on exams.

Pictures of Numbers: The Number Line

Example 1

Find the interval that corresponds to the graph.



A. $(7, -5]$

B. $[-5, 7)$

C. $[7, -5)$

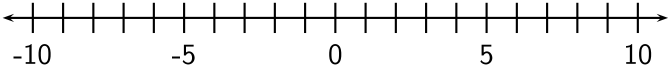
D. $(-5, 7]$

E. $(-5, 7)$

Pictures of Numbers: The Number Line

Example 2

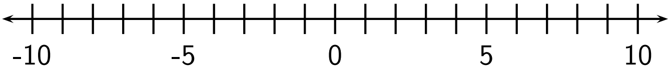
Graph the interval $(-2, \infty)$ on a number line.



Pictures of Numbers: The Number Line

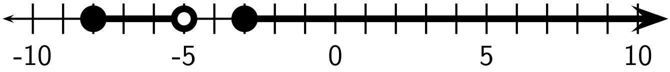
Example 3

Graph the interval $(-\infty, -2]$ on a number line.



Pictures of Numbers: The Number Line

If you need to include values that are in one interval OR another, we use the union operator. For example, the interval notation for

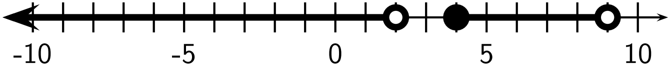


is $[-8, -5) \cup [-3, \infty)$.

Pictures of Numbers: The Number Line

Example 4 (Do you understand ∪?)

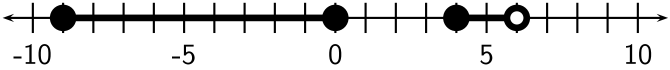
Write the interval notation that corresponds to the following graph?



Pictures of Numbers: The Number Line

Example 5

Find the interval that corresponds to the graph.



A. $[0, -9] \cup (6, 4]$

B. $[0, -9] \cup [4, 6)$

C. $[-9, 0] \cup [4, 6)$

D. $(-10, 1) \cup (3, 6)$

E. $(-9, 0) \cup (4, 6]$

Distance on the Number Line: Absolute Value

Definition 6 (Absolute Value - Geometric Defintion)

The **absolute value** of a number x, denoted $|x|$, is the distance between x and 0 on a number line.

Distance on the Number Line: Absolute Value

Example 7

Draw a picture using the number line that represents the definition of $|x|$ when

1. x is positive.

2. x is negative.

Distance on the Number Line: Absolute Value

If x is non-negative, then $|x| =$ ______________.

If x is negative, then $|x| =$ ______________.

Distance on the Number Line: Absolute Value

Definition 8 (Absolute Value - Algebraic Defintion)

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Distance on the Number Line: Absolute Value

Example 9 (Do you understand Absolute Value?)

1. $|5.7| =$
2. $|-\pi| =$
3. $|6-\pi| =$
4. $|2-\pi| =$

Distance on the Number Line: Absolute Value

Several properties of the absolute value function are covered on page 10 of your textbook. **You are responsible** for reviewing these properties.

Distance on the Number Line: Absolute Value

Two special properties that you may not recall are below.

Property 10

If c is a real number, then $\sqrt{c^2} = |c|$.

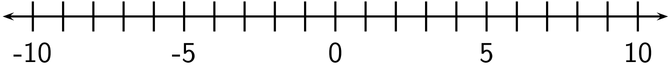
Property 11

If x and y are real numbers then $|x - y| = |y - x|$

Distance on the Number Line: Absolute Value

Example 12

Find the distance between -2 and 5 on the number line. Can you write two different expressions for this distance using absolute value notation?



Distance on the Number Line: Absolute Value

Definition 13

The **distance between** x **and** y **on the number line** is

Solving Equations with One Variable Type - The Algebraic Approach

When two expressions are set equal to each other, the result is an **equation**. **Equations contain an equals sign. Expressions do not.** Equations may or may not contain variables. A **solution** to an equation is any substitution for the variables in an equation that results in a true mathematical statement.

Solving Equations with One Variable Type - The Algebraic Approach

Example 14 (Solutions to Equations)

Which of the following is a solution to $3 - 5x = 2(4 - x) + 1$?

$x = -2$

$x = 4$

Solving Equations with One Variable Type - The Algebraic Approach

Consider the following equations:

$$x + 2 = 4$$

$$5x = 6 + 2x$$

$$3x^3 = 24$$

The only substitution for x that results in a true statement in each of these equations is $x = 2$. This means 2 is the only solution to each equation.

Solving Equations with One Variable Type - The Algebraic Approach

Equation	Variable Types
$5x = 6 + 2x$	x
$3x^3 = 24$	x^3
$2\sqrt{x} = 7$	$\sqrt{x}$
$3x^2 = 2x + 1$	x, x^2

Equivalent Equations

Definition 15

Two equations are **equivalent** if they have the same solutions.

For example,

$$2x + 5 = 3x - 1,$$

$$5 = x - 1, \text{ and}$$

$$6 = x$$

are equivalent equations since $x = 6$ is the only solution for ALL THREE equations.

Equivalent Equations

Operations that Produce an Equivalent Equation:

1. Add or subtract the same number to both sides of the equation.
2. Add or subtract the same algebraic expression that is always defined to both sides of the equation.
3. Multiply or divide both sides of the equation by a **NONZERO** number.
4. Add zero to one side of the equation.
5. Multiply one side of the equation by 1.

Equivalent Equations

Example 16 (Equivalent Equations)

In each of the following cases, decide if the action always, sometimes or never produces an equivalent equation. Justify your answer.

- Squaring both sides of an equation

- Adding x to both sides of an equation

- Multiplying both sides of an equation by x

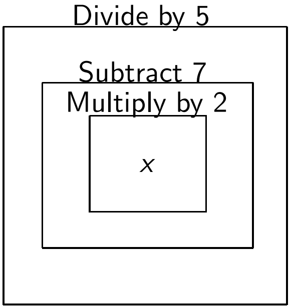
Equivalent Equations

Ideally, we would like to keep equivalent equations as we move from one step to the next in the solution. This is not always possible. Sometimes you need to square both sides of the equation. Sometimes you need to multiply both sides of an equation by an algebraic expression instead of a number. These operations can produce **extraneous solutions**. This is why it is important to **CHECK YOUR SOLUTIONS**.

Unwrapping a Variable

$$\frac{2x-7}{5}=8.$$

What operations are being applied to x? In what order are these operations applied?



Unwrapping a Variable

$$\frac{2x-7}{5} = 8.$$

To solve this equation we should do the following:

Solve the equation.

Unwrapping a Variable

$$3\left(\frac{2-s}{8}\right)=5$$

What operations are being applied to s? In what order are these operations applied?

> It is easier to solve equations if you think of subtraction as adding a negative number.

Unwrapping a Variable

Example 17

Solve for s.

$$3\left(\frac{2-s}{8}\right)=5$$

Unwrapping a Variable

Example 18

Solve for s.

$$3\left(\frac{2-s}{8}\right)=\frac{s+6}{12}$$

Unwrapping a Variable

Example 19

Solve for t.

$$2(3t+1)-7=5$$

Unwrapping a Variable

Example 20

Solve for r.

$$C = 2\pi r$$

Unwrapping a Variable

Example 21 (Concept Check)

To solve for b in the equation below, what should you do first?

$$ax + b^2y = 1$$

A. Divide both sides by a.

B. Subtract ax from both sides.

C. Take the square root of both sides.

D. Divide both sides by y.

Solving Fractional Equations

When an equation has a variable in a denominator:

1. Find a common multiple for all denominators in the equation.
2. Multiply both sides of the equation by the common multiple.
3. Solve the new equation. *Be CAREFUL! The new equation may not be equivalent to the original equation. You may find some* **extraneous solutions** *when you solve the new equation.*
4. Check all of your solutions in the original equation. Keep only those solutions that are solutions of the original equation.

Solving Fractional Equations

Example 22

Solve for y.

$$\frac{y}{y+1} = \frac{1}{y^2+y}$$

Solving Fractional Equations

Example 23

Solve for u.

$$\frac{1}{F} = \frac{1}{v} + \frac{1}{u}$$

Solving Power Equations

Solutions of Power Equations

The real solution(s) of $x^n = a$ is(are):

- $x = \sqrt[n]{a}$ if n is odd
- $x = \sqrt[n]{a}$ and $x = -\sqrt[n]{a}$ if n is even and $a \geq 0$

If n is even and $a < 0$, then $x^n = a$ does not have any real solutions.

Solving Power Equations

Example 24

Solve.

(a) $x^4 = 10$

Solving Power Equations

(b) $\dfrac{x^3 + 5}{2} = 1$

Solving Power Equations

(c) $3(x-4)^2+1=7$

Solving Power Equations

Example 25

Solve for r.

$$A = \pi r^2$$

A Return to Absolute Value

Solutions of Absolute Value Equations
The real solution(s) of $|x| = a$ are $x = a$ and $x = -a$ if $a \geq 0$.

A Return to Absolute Value

Example 26 (Distance Example)

Solve $|x - 3| = 4$ geometrically.

A Return to Absolute Value

Example 27 (Another Distance Example)

Solve $|4 - x| + 3 = 11$.

A Return to Absolute Value

Example 28

Solve $|3x + 2| + 1 = 5$

Quadratic Equations

Definition 29

A **quadratic equation in** x is any equation that is equivalent to an equation of the form $ax^2 + bx + c = 0$ with $a \neq 0$.

Quadratic Equations

$2x^2 + 3x + 5 = 0$ is a quadratic equation in x.
$6u + 5u^2 = 2$ is a quadratic equation in u.
$\dfrac{4z^2 + 2}{5} = 7$ is a quadratic equation in z.

$2x + 3 = 0$ is not a quadratic equation.
$\dfrac{1}{x} + x^2 - 2 = 0$ is not a quadratic equation.

Quadratic Equations

Property 30 (Zero Product Property)

If $AB = 0$ *then* $A = 0$ *or* $B = 0$

Quadratic Equations

Example 31

Use the Zero Product Property to solve $x^2 - 9x = -20$.

Quadratic Equations

Example 32

Use the Zero Product Property to solve $x^2 + 5x - 6 = 0$.

Quadratic Equations

Example 33

Use the Zero Product Property to solve $3x^2 - 5x + 2 = 0$.

Quadratic Equations

Example 34 (A Factoring Example)

Factor the following expressions.

- $x^2 + 6x + 9$
- $x^2 - 8x + 16$
- $x^2 - 7x + \frac{49}{4}$

Quadratic Equations

Example 35 (Completing The Square)

Fill in the blank so that the following will factor as a perfect square.

$$x^2 - 12x + \underline{\qquad\qquad}$$

(a) 24
(b) 6
(c) -6
(d) 36
(e) -36
(f) 144

Quadratic Equations

Example 36 (Completing The Square)

Solve $x^2 + 10x + 4 = 0$ by completing the square.

Quadratic Equations

Example 37 (Completing The Square)

Solve $x^2 - 6x + 11 = 0$.

Quadratic Equations

Example 38 (Completing the Square)

Solve $2x^2 - 8x + 1 = 0$ by completing the square.

Quadratic Equations

Each time you complete the square, you are going through the exact same process. You could start with a generic quadratic equation $ax^2 + bx + c = 0$ $(a \neq 0)$ and complete the square with it. (This is done for you in your textbook.) Upon completing the square, you produce a formula for the solution(s).

Theorem 39 (The Quadratic Formula)

The solutions of $ax^2 + bx + c = 0$ $(a \neq 0)$ *are*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Although memorizing is not the best strategy in mathematics, your life will be easier if you memorize the quadratic formula.

Quadratic Equations

Example 40

Use the Quadratic Formula to solve $2x^2 - 8x + 1 = 0$.

Quadratic Equations

Example 41

Use the Quadratic Formula to solve $3x^2 - 5x + 2 = 0$.

Quadratic Equations

Definition 42

In the quadratic formula, the expression $b^2 - 4ac$ is called the **discriminant**.

Quadratic Equations

Theorem 43 (Number of Real Solutions of a Quadratic Equation)

If the discriminant of a quadratic equation is positive, the equation has two solutions. If it is zero, the equation has one solution. If it is negative, the equation does not have any real solutions.

Quadratic Equations

Example 44 (The Discriminant)

How many real solutions does each equation have?

(a) $3x^2 + 2x + 5 = 0$

(b) $x^2 + 5x = 7$

(c) $2x^2 = 12x - 18$

Quadratic Type Equations

Some equations have the form $au^2 + bu + c = 0$ where u is an algebraic expression. We call these equations **quadratic type equations.**

To solve quadratic type equations:

1. Look for an expression and its square.
2. Let u be the expression.
3. Substitute u for the expression and u^2 for the square of the expression. The only variable in the new equation should be u. None of the original variables should remain.
4. Solve the new equation for u.
5. In the solution of the new equation, substitute the original expression for u. This will contain the original variable.
6. Solve for the original variable.
7. CHECK YOUR SOLUTIONS!

Quadratic Type Equations

Example 45

Solve for x.

$$x^4 - 2x^2 - 3 = 0$$

Quadratic Type Equations

Example 46

Solve for t.

$$2t^{1/6} + 8 = t^{1/3}$$

Quadratic Type Equations

Example 47

Solve for z.

$$\frac{1}{(z+1)^2} - 3 = \frac{2}{z+1}$$

Other Types of Equations

Example 48

Find all real solutions to the equation.

$$4x^4 = 16x^2$$

Other Types of Equations

Example 49

Find all real solutions to the equation.

$$7x^3 + 3x^2 - 3 = 10x - 3x^3$$

Other Types of Equations

The last type of equation we will solve in this section is equations with radicals. Previously, we stated that squaring both sides of an equation does not necessarily produce an equivalent equation. However, it may be necessary for equations which involve radicals. This may create extraneous solutions. **If you square both sides of an equation, you must check your answers.**

Example 50

Find all real solutions to the equation.

$$\sqrt{1-t} = t + 5$$

Other Types of Equations

Example 51

Find all real solutions to the equation.

$$2 + \sqrt{a} = a$$