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(Chapter 5)

Rules of Exponents

The following are to remind you of the rules of exponents. You are expected to know how to use them. To review, see section 5.1 in your textbook.

Let c be a nonnegative real number, and let r and s be any rational numbers. Then

$$c^r c^s = c^{r+s} \qquad \frac{c^r}{c^s} = c^{r-s},\ (c \neq 0)$$

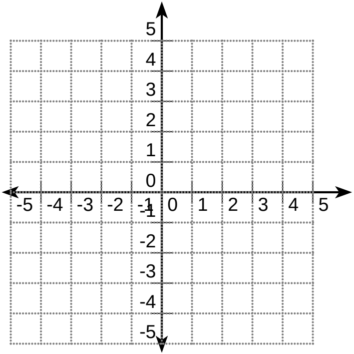
$$c^{-r} = \frac{1}{c^r},\ (c \neq 0) \qquad (c^r)^s = c^{rs}$$

$$c^{1/s} = \sqrt[s]{c} \qquad c^{r/s} = \sqrt[s]{c^r} = \left(\sqrt[s]{c}\right)^r$$

Power Functions vs. Exponential Functions

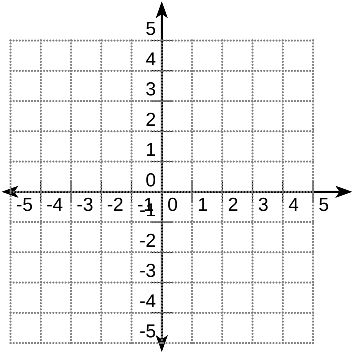
Example 1

- Sketch the graphs of $y = P(x) = x^2$ and $y = E(x) = 2^x$ on the same graph.



Power Functions vs. Exponential Functions

- Sketch the graphs of $y = P(x) = x^3$ and $y = E(x) = 3^x$ on the same graph.



Power Functions vs. Exponential Functions

In the previous example, both of the P functions are *power functions*, and both of the E functions are *exponential functions*.

- What are the characteristics of a power function?

- What are the characteristics of an exponential function?

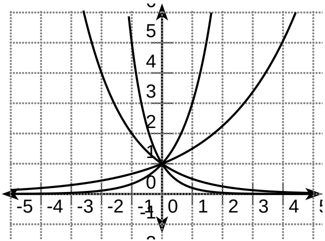
Power Functions vs. Exponential Functions

We can see that power functions are very different than exponential functions, so we should expect to treat them in very different ways. Solving a power equation is very different than solving an exponential equation. Finding the inverse function (if there is one) of a power function is very different than finding an inverse function of an exponential function. Do not be confused because both types of functions have exponents. **It matters if the variable is in the base or in the exponent.**

The Graphs of Exponential Functions

Example 2 (Exponential Graphs)

The graphs of $y = f(x) = 3^x$, $y = g(x) = 1.5^x$, $y = h(x) = 0.5^x$ and $y = k(x) = 0.2^x$ are drawn below for you. Label the graphs with their function names. Compare and contrast the graphs.



The Graphs of Exponential Functions

What are some characteristics of the graph of $y = f(x) = a^x$ if $a > 1$?

What are some characteristics of the graph of $y = f(x) = a^x$ if $0 < a < 1$?

The Graphs of Exponential Functions

What happens if you try to graph $y = b(x) = (-2)^x$?

What happens if you try to graph $y = c(x) = 1^x$?

What happens if you try to graph $y = d(x) = 0^x$?

Understanding Exponential Functions

Definition 3 (Exponential Functions)

Let a be a positive number that is not equal to one. The **exponential function** with base a is a function that is equivalent to $f(x) = a^x$.

NOTE: Your textbook does not tell you that $a \neq 1$. However, because this function behaves so differently when $a = 1$, most textbooks do not call $g(x) = 1^x$ an exponential function. In this course, we will follow the convention that $g(x) = 1^x$ is **NOT** an exponential function.

Understanding Exponential Functions

Example 4 (Understanding Exponential Growth)

Suppose that you place a bacterium in a jar. Each bacterium divides into 2 bacteria every hour.

- How many bacteria are in the jar after 2 hours?
- How many bacteria are in the jar after 4 hours?
- How many bacteria are in the jar after 9 hours?
- Write a function to express the total number, P, of bacteria are in the jar after t hours?

Understanding Exponential Functions

- At the 30 hour mark, the jar was completely full. How many bacteria were in the jar at this time?

- When was the jar half full?

Understanding Exponential Functions

Lots of quantities grow by a certain multiple. For example, a bacteria population model may claim that the bacteria population is doubling every hour (as in the previous example). These situations yield what is known as **exponential growth models**.

Understanding Exponential Functions

Definition 5 (Exponential Growth)

If a quantity, P, can be modeled by a function of the form

$$P(t) = P_0 a^t,$$

where $a > 1$ and t represents time, then P is said to **grow exponentially**.

Notice that P_0 is the initial amount of the quantity because

____________________________.

Understanding Exponential Functions

Example 6

A bacteria culture starts out with 1000 bacteria and doubles every 3 hours. How many bacteria will there be after 5 hours?

Understanding Exponential Functions

There is a similar phenomenon called **exponential decay**. This occurs when $0 < a < 1$.

Definition 7 (Exponential Decay)

If a quantity, Q, can be modeled by a function of the form

$$Q(t) = Q_0 a^t$$

where $0 < a < 1$ and t represents time, then Q is said to **decay exponentially**.

Exponential growth and exponential decay are, for all practical purposes, the same idea.

Understanding Exponential Functions

Example 8 (Radioactive Decay)

The half life of Actinium-225 (Ac-225) is 10 days. How much of a 30-gram sample of Ac-225 is left after one year?

Understanding Exponential Functions

Example 9

JB Outlet Store is having a sale on tents. Every day a tent does not sell, its price is marked down 20%. If the price of a tent is \$100 on Sunday, what is the price of the tent on Friday if it has not sold?

Compound Interest

Exponential growth could also be described as a certain percentage rate which is compounded. When the quantity is increasing by the rate r (as a decimal) for every unit of time, then $a = 1 + r$, so that the model can be written as

$$P(t) = P_0(1 + r)^t.$$

Compound Interest

A special case of exponential growth that you are sure to run into is compound interest. When the interest is compounded yearly, there is no ambiguity. However, many times interest is compounded semiannually, quarterly, monthly, or even weekly. The interest rate is still given as an *annual* interest and time is usually given in years.

In these cases, we must introduce another variable n which is the number of times per year the interest is compounded.

Compound Interest

Proposition 10 (Compound Interest)

If a principal P_0 is invested at an annual interest rate r for a period of t years, then the amount $P(t)$ of the investment is given by:

$$P(t) = P_0\left(1 + \frac{r}{n}\right)^{nt} \quad \text{(if compounded } n \text{ times per year)}$$

Compound Interest

Example 11 (Understanding Exponential Growth)

Suppose you invest \$10,000 in an account that earns 5% interest compounded semiannually.

- Write a function that expresses the amount of money in the account after t years.
- How much money will you have in 8 years?

Compound Interest

Example 12 (Understanding Exponential Growth)

Suppose you invest \$12,000 in an account that earns 3% interest compounded quarterly. How much money will you have in 1 year?

Compound Interest

e is a special irrational number for a multitude of reasons that you will only begin to understand in Calculus.

It is important to note that the number e is a number like π is a number. It is not a variable. It is an irrational number. You never can have an exact value for e. The best you can hope to have is a decimal approximation. $e \approx 2.71828182845....$

We can use compound interest to begin to explain the value of e.

Compound Interest

Example 13 (The number e)

Suppose that you invest \$1 at an annual interest rate of 100% compounded annually. How much money will you have after 1 year?

Compound Interest

Suppose that you invest \$1 at an annual interest rate of 100% compounded monthly. How much money will you have after 1 year?

Compound Interest

Suppose that you invest \$1 at an annual interest rate of 100% compounded daily. How much money will you have after 1 year?

Compound Interest

Suppose that you invest $1 at an annual interest rate of 100% compounded every minute. How much money will you have after 1 year?

Compound Interest

What does the value $\left(1+\frac{1}{n}\right)^n$ seem to be approaching as n becomes large?

Compound Interest

Suppose that you invest \$1 at an annual interest rate of 100% that could be compounded continuously. How much money should you expect to have after 1 year?

Compound Interest

In Calculus, you will see that as n becomes very large, the quantity $\left(1+\frac{1}{n}\right)^n$ approaches the value e. This fact can be used to justify the following formula for continuously compounded interest.

Proposition 14 (Continuous Compounding)

If P_0 dollars is invested at an annual interest rate r (as decimal), compounded continuously, then the value of the investment after t years is given by

$$P(t) = P_0 e^{rt}.$$

Compound Interest

Example 15 (Continuously Compounded Interest)

Jake invests \$1000 at an annual interest rate of 4.6% compounded continuously. How much money will Jake have in 15 years?

Logarithms

Exponential functions are one-to-one functions. Consequently, each exponential function has an inverse function. Why might you want to undo exponentiation? Suppose you want to solve the following equation.

$$10^x = 3$$

What is happening to x? ______________________________

Logarithms

How do we undo this?

$$10^x = 3$$

Logarithms with base a

The logarithm with base a is the inverse function of $f(x) = a^x$. The name of the logarithm with base a function is $\log_a$ (said "log base a").

Definition 16 (Logarithms with Base a)

Let x and y be real numbers with $x > 0$. Let a be a positive real number that is not equal to 1. Then

$$\log_a(x) = y \text{ if and only if } a^y = x.$$

In other words, the $\log_a(x)$ picks the exponent to which a must be raised to produce x.

Logarithms with base a

There are a few special logarithms with special names. The logarithm with base 10 is most often called the **common logarithm** is written $\log(x)$. The logarithm with base e is most often called the **natural logarithm** and is written $\ln(x)$.

Logarithms with base *a*

Example 17 (Exponential Notation and Logarithmic Notation)

Convert the exponential statement to a logarithmic statement.

$$5^3 = 125$$

$$10^{-3} = \frac{1}{1,000}$$

$$e^2 \approx 7.389$$

Logarithms with base *a*

Example 18

Convert the logarithmic statement to an exponential statement.

$$\log_3(3^5) = 5$$

$$\log(10{,}000) = 4$$

$$\ln(1) = 0$$

Logarithms with base a

Example 19

Evaluate each of the following.

$\log(100)$

$\log\left(\frac{1}{1000}\right)$

$\log_2(16)$

$\ln\left(\frac{1}{\sqrt{e}}\right)$

$\log_{\frac{1}{2}}(16)$

$\log(10^9)$

$\log(\sqrt[3]{100})$

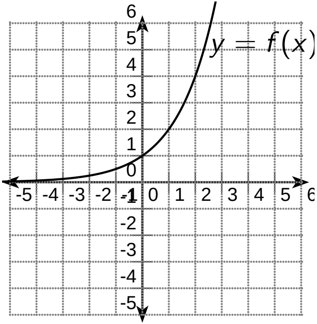
$\ln(e^5)$

$\log_3\left(\frac{1}{81}\right)$

Logarithms with base a

Example 20 (Logarithm with base 2 Graph)

- The graph of $y = f(x) = 2^x$ is drawn below. Sketch the graph of $y = g(x) = \log_2(x)$ on the same coordinate system.
- What is the domain of $g(x) = \log_2(x)$? What is the range of $g(x) = \log_2(x)$?



Logarithms with base a

What are some characteristics of the graph of $y = f(x) = \log_a(x)$ if $a > 1$?

What are some characteristics of the graph of $y = f(x) = \log_a(x)$ if $0 < a < 1$?

Logarithms with base a

Notice that the input of a logarithm must be greater than 0. This is one more function where you must be careful when finding the domain.

Logarithms with base a

Example 21 (Logarithm Domain)

Find the domain of $f(x) = \log_7(2 - 5x)$.

Logarithms with base a

Example 22 (Logarithm Domain)

Find the domain of $f(x) = \ln(x^2 - 4x + 3)$.

Properties of Logarithms

Each property of logarithms is derived from the definition of the logarithm and/or a property of exponents.

Property 23

- $\log_a(1) =$ ______________
- $\log_a(a) =$ ______________
- $\log_a(a^x) =$ ______________
- $a^{\log_a(x)} =$ ______________

Notice that all properties can be stated in terms of the $\log_a(x)$ function since $\ln(x) = \log_e(x)$ and $\log(x) = \log_{10}(x)$

Properties of Logarithms

Example 24

- Simplify $e^{x \ln(2)}$.

- Rewrite 5^x as e to a power.

Properties of Logarithms

Example 25

Evaluate $\log(10^5 * 10^3)$.

Properties of Logarithms

Property 26 (Product Law for Logarithms)

For all $u > 0$ *and* $v > 0$

- $\log_a(uv) = \log_a(u) + \log_a(v)$

Proof:

Properties of Logarithms

Property 27 (Quotient Law for Logarithms)

For all $u > 0$ *and* $v > 0$

- $\log_a\left(\frac{u}{v}\right) = \log_a(u) - \log_a(v)$

Properties of Logarithms

Example 28

Use the properties of logarithms to express $\ln\left(\frac{xy}{z}\right)$ as a sum and or difference of three logarithms.

Properties of Logarithms

Example 29

Use the properties of logarithms to write the expression using the fewest number of logarithms possible.

$$\log(x^2 + 2) + \log(x) - \log(y) - \log(z)$$

Properties of Logarithms

Property 30 (Power Law for Logarithms)

For all $u > 0$ *and all* k

- $\log_a(u^k) = k \log_a(u)$

Proof:

Properties of Logarithms

Example 31

Use the properties of logarithms to express $\log_5\left(\frac{x^3}{y\sqrt{z}}\right)$ in terms of $\log_5(x)$, $\log_5(y)$, and $\log_5(z)$.

Properties of Logarithms

Example 32

Use the properties of logarithms to write the expression using the fewest number of logarithms possible.

$$\ln(x^2) - 2\ln(y) - 3\ln(z)$$

Properties of Logarithms

Property 33 (Change of Base)

If $a, b, x > 0$ and neither a nor b equals 1, then

$$\log_a(x) = \frac{\log_b(x)}{\log_b(a)}.$$

Properties of Logarithms

Example 34 (Change of Base)

Use your calculator to approximate $\log_5(67)$.

Solving Exponential and Logarithmic Equations

Remember, "undoing" an operation means applying the inverse operation to both sides of an equation. The exponential function a^x and the logarithmic function $\log_a(x)$ are inverses of each other.

Solving Exponential and Logarithmic Equations

Example 35

Solve.

$$\log(x+5) = 3$$

Solving Exponential and Logarithmic Equations

Example 36

Solve.

$$\log_8(x-5)+\log_8(x+2)=1$$

Solving Exponential and Logarithmic Equations

The properties of logarithms only work when the input is positive. If you use them to solve an equation involving logarithms, **you must check your answer(s)**.

Solving Exponential and Logarithmic Equations

Example 37

Solve.

$$e^{x+2} = 5$$

Solving Exponential and Logarithmic Equations

Example 38

Solve.

$$\frac{2^x - 7}{3} = -1$$

Solving Exponential and Logarithmic Equations

Example 39

Solve.

$$2^{x-5} = 3^{2-2x}$$

Solving Exponential and Logarithmic Equations

Example 40

Joni invests \$1000 at an interest rate of 5% compounded monthly. When will the value of Joni's investment reach \$2500?

Solving Exponential and Logarithmic Equations

Example 41

A bacteria culture triples every 4 hours. How long until the culture doubles?