# 12 Rational Functions

**Concepts:**

- The Definition of a Rational Function
- Identifying Rational Functions
- Finding the Domain of a Rational Function
- The Big-Little Principle
- Vertical and Horizontal Asymptotes
- The Graphs of Rational Functions

**(Section 4.5)**

**Definition 12.1**
A **rational function** is a function that is equivalent to a function of the following form.

$$r(x) = \frac{\text{polynomial}}{\text{polynomial}}$$

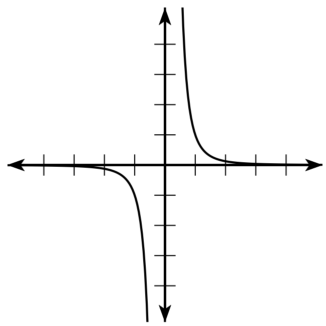
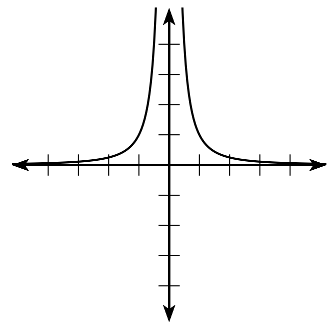
**Example 12.2**
Which of the following are rational functions? If the function is rational, find its domain.

- $f(x) = \dfrac{x^2 + 2x + 1}{x - 3}$ Rational
  $f(x)$ is defined if $x \neq 3$, so the domain of $f(x)$ is $(-\infty, 3) \cup (3, \infty)$
- $g(x) = \dfrac{2\sqrt{x+1}}{x^2 + 2x}$ Not rational since the numerator is not a polynomial.
- $h(x) = \dfrac{\sqrt{6} + 3x}{x^5}$ Rational
  $h(x)$ is defined for all $x$ except 0, so the domain of $h(x)$ is $(-\infty, 0) \cup (0, \infty)$.
- $j(x) = \dfrac{x+1}{x-2} + x = \dfrac{x+1}{x-2} + \dfrac{x(x-2)}{x-2} = \dfrac{x + 1 + x^2 - 2x}{x-2} = \dfrac{x^2 - x + 1}{x - 2}$ Rational
  $j(x)$ is defined if $x \neq 2$, so the domain of $j(x)$ is $(-\infty, 2) \cup (2, \infty)$.
- $k(x) = x^5 + 2x + 1 = \dfrac{x^5 + 2x + 1}{1}$ Rational
  $k(x)$ is defined for all $x$ so its domain is $(-\infty, \infty)$.

**Note 12.3**
Polynomials are a special case of rational functions.

## 12.1 Graphs of Some Simple Rational Functions

Let $f(x) = \frac{1}{x^n}$. Below is a chart of the basic shapes that the graph of $f(x)$ can take on.

| $n$ odd | $n$ even |
|---|---|
|  |  |
| Examples: $f(x) = \frac{1}{x^5}$, $g(x) = \frac{1}{x^7}$ | Example: $f(x) = \frac{1}{x^6}$, $h(x) = \frac{1}{x^{10}}$ |

**Property 12.4 (The Big-Little Principle)**
*If $c$ is a number that is close to 0 on the number line, then $\frac{1}{c}$ is a number that is far from 0 on the number line.*

*If $c$ is a number that is far from 0 on the number line, then $\frac{1}{c}$ is a number that is close to 0 on the number line.*

How should you remember the Big-Little Principle?
If $c$ is big then $\frac{1}{c}$ is little and if $\frac{1}{c}$ is big then $c$ is little.

Can you use the Big-Little Principle to explain the shape of the graph of $y = f(x) = \frac{1}{x^n}$?
How do you describe the end behavior of the graph of $y = f(x) = \frac{1}{x^n}$?

If $x$ is big (in either the positive or negative sense) then $\frac{1}{x}$ is little so $\frac{1}{x^n}$ is also little. As $x$ gets bigger, $\frac{1}{x^n}$ gets littler - that is, it gets closer to zero. So, the end behavior of $f(x)$ is

$$y \to 0 \text{ as } x \to \infty$$

$$y \to 0 \text{ as } x \to -\infty$$

Many (though not all) of the graphs of rational functions have **asymptotes**. Intuitively, if a graph approaches another graph and eventually gets as close to that other graph as anyone could possibly hope without necessarily touching the other graph, then the other graph is called an asymptote for the original graph. Two types of asymptotes that often occur in the graphs of rational functions are **horizontal asymptotes** (asymptotes that are horizontal lines) and **vertical asymptotes** (asymptotes that are vertical lines).

You can determine a rational function's **horizontal asymptotes** by considering the leading terms of the numerator and denominator. (This is concept is very similar to the end behavior of a polynomial.) You can determine a rational function's **vertical asymptotes** by finding the $x$ values which are zeros of the denominator but are not zeros of the numerator.

**Example 12.5**
Describe the horizontal asymptotes of the graph of

$$y = g(x) = \frac{3x^2 + 2x + 5}{5x^3 + 2x^2 + 7x + 1}.$$

The leading term of the numerator is $3x^2$ and the leading term of the denominator is $5x^3$. The ratio of the leading terms is $\frac{3x^2}{5x^3} = \frac{3}{x}$. Thus, $g(x)$ has the same end behavior as $\frac{1}{x}$ which is

$$y \to 0 \text{ as } x \to \infty \qquad \text{and} \qquad y \to 0 \text{ as } x \to -\infty$$

So, $y = 0$ is a horizontal asymptote.

**Example 12.6**
Describe the horizontal and vertical asymptotes of the graph of

$$y = f(x) = \frac{2x + 1}{x - 3}.$$

The leading term of the numerator is $2x$ and the leading term of the denominator is $x$. The ratio of the leading terms is $\frac{2x}{x} = 2$. Thus, $f(x)$ has the same end behavior as $y = 2$ which is

$$y \to 2 \text{ as } x \to \infty \qquad \text{and} \qquad y \to 2 \text{ as } x \to -\infty$$

So, $y = 2$ is a horizontal asymptote. $x = 3$ is a zero of the denominator but not the numerator. So $x = 3$ is a vertical asymptote of $f(x)$.

**Example 12.7**
Describe the horizontal and vertical asymptotes of the graph of

$$y = g(x) = \frac{x^2 + 5x + 6}{x + 4}.$$

The leading term of the numerator is $x^2$ and the leading term of the denominator is $x$. The ratio of the leading terms is $\frac{x^2}{x} = x$. Thus, $g(x)$ has the same end behavior as $y = x$ which is

$$y \to \infty \text{ as } x \to \infty \qquad \text{and} \qquad y \to -\infty \text{ as } x \to -\infty$$

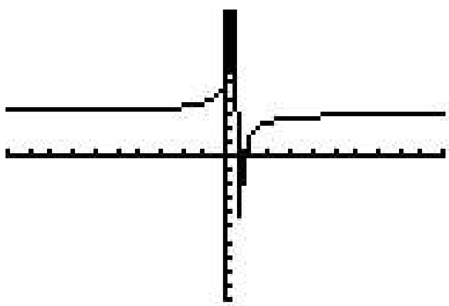
So, $g(x)$ does not have a horizontal asymptote. Notice that the numerator $x^2+5x+6 = (x+3)(x+2)$. So, $x = -4$ is a zero of the denominator but not the numerator. So $x = -4$ is a vertical asymptote of $g(x)$.

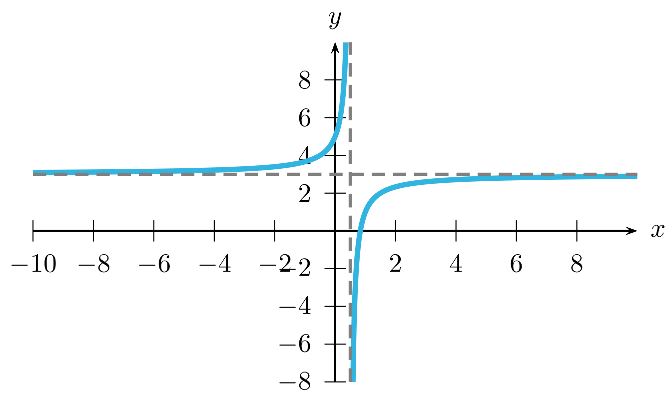
## 12.2 The Graphs of Rational Functions - Some Examples

We have used a graphing calculator (TI-83 Plus) to approximate the graphs of a few rational functions. Graphing calculators do not do a very good job of sketching asymptotes. Nevertheless, we can use them to better understand the graphs of rational functions. For each graph, look at the algebraic description of the function and the approximate graph to better understand its asymptotes. Show algebraically how you would find the asymptotes of each graph. Draw a better sketch of the graph that includes all asymptotes. Make sure that **asymptotes are drawn with dotted lines**.

**Note:** *Each graph is in a* $[-10, 10] \times [-10, 10]$ *viewing window.*

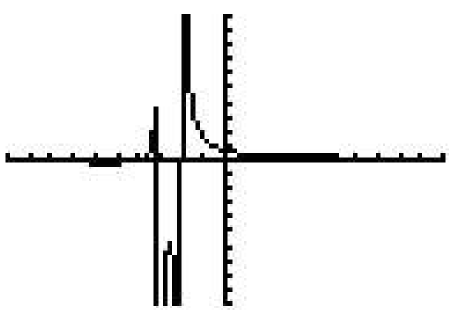
- The graph of $y = f(x) = \dfrac{6x-5}{2x-1}$.

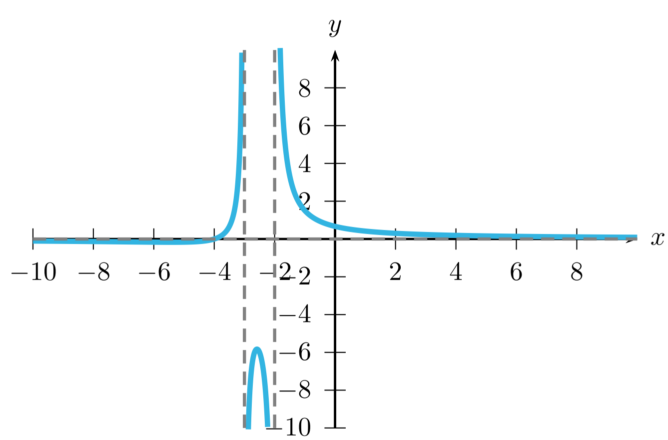





The leading term of the numerator is $6x$ and the leading term of the denominator is $2x$. The ratio of the leading terms is $\dfrac{6x}{2x} = 3$. Thus, $f(x)$ has the same end behavior as $y = 3$ which is $y \to 3$ as $x \to \infty$ and$y \to 1$ as $x \to -\infty$ So, $y = 3$ is a horizontal asymptote of $f(x)$. Notice that $x = \dfrac{1}{2}$ is a zero of the denominator but not the numerator. So $x = \dfrac{1}{2}$ is a vertical asymptote of $f(x)$.

- The graph of $y = f(x) = \dfrac{x+4}{x^2+5x+6}$.






The leading term of the numerator is $x$ and the leading term of the denominator is $x^2$. The ratio of the leading terms is $\dfrac{x}{x^2} = \dfrac{1}{x}$. Thus, $f(x)$ has the same end behavior as $y = \dfrac{1}{x}$ which is $y \to 0$ as $x \to \infty$ and $y \to 0$ as $x \to -\infty$ So, $y = 0$ is a horizontal asymptote of $g(x)$. The denominator $x^2 + 5x + 6 = (x+3)(x+2)$ is zero when $x = -3$ and when $x = -2$. Since neither of these make the numerator zero then $x = -3$ and $x = -2$ are vertical asymptotes of $f(x)$.

**Example 12.8 (Graphs of Rational Functions)**

Let $h(x) = \dfrac{x^2 + x - 2}{2x^2 - 8x - 10}$. Sketch the graph of $h(x)$ without using your calculator. Be sure to label all asymptotes and intercepts of the graph.

The leading term of the numerator is $x^2$ and the leading term of the denominator is $2x^2$. The ratio of the leading terms is $\dfrac{x^2}{2x^2} = \dfrac{1}{2}$. Thus, $h(x)$ has the same end behavior as $y = \dfrac{1}{2}$ which is

$$y \to \frac{1}{2} \text{ as } x \to \infty \qquad \text{and} \qquad y \to \frac{1}{2} \text{ as } x \to -\infty$$

So, $y = \dfrac{1}{2}$ is a horizontal asymptote. Notice that the numerator $x^2 + x - 2 = (x+2)(x-1)$ and the denominator $2x^2 - 8x - 10 = 2(x+1)(x-5)$ . So, $x = -1$ and $x = 5$ are zeros of the denominator but not the numerator. So $x = -1$ and $x = 5$ are vertical asymptotes of $h(x)$.

The $y$-intercept of the graph is $y = \dfrac{0^2 + 0 - 2}{2(0)^2 - 8(0) - 10} = \dfrac{-2}{-10} = \dfrac{1}{5}$. The $x$-intercepts are found by setting $h(x) = 0$. The $x$-intercepts are where the numerator of $h(x)$ are 0. Since we've already factored the numerator above, its easy to see that the $x$-intercepts are $x = -2$ and $x = 1$.